CHAPTER 18
LOGICAL PARADOXES

18.1. Logical paradoxes are a challenge to the human mind. A challenge I presume to have won.

Along the lines of the distinction proposed by Ramsey in *The foundations of mathematics*, today there is a tendency to classify as logical only the paradoxes which can be formulated in the language of a formal theory. Thus the father Liar, but also Grelling and so on are classified as epistemological paradoxes. In my opinion the analogies between the two classes are too profound to make momentous such a distinction. I join then Goddard and Johnston where they claim (1983, p.491) that *what is important about the paradoxes is their common feature, not their differences ...in this sense all are logical*). Of course "all" must be interpreted in the nearly obvious acceptation according to which the quantification does not concern absolutely distinct disciplines (as for instance hydrostatics (D'Alembert) or probability (Bértrand). On the other hand I am ready to classify among logical paradoxes results that today are not considered as paradoxes (Schoenfinkel's reduction), and even results that today are considered supreme achievements (Cantor's proof, Gödel's incompleteness theorems).

18.2. I agree with Nelson-Lesniewski's claim (Sobocinski, 1949) according to which a paradox is much more than a contradiction: it is a contradiction derived through a seemingly unexceptionable argument from seemingly unexceptionable premises. Consequently every procedure that forbids the derivation of the paradox without explaining clearly the logical mistake it is born by, represents only a sheer expedient; for instance it is a sheer expedient to banish some intuitively adequate rule of transformation or to mutilate the language in order to forbid the very possibility of formulating the paradox. Since I join Goddard and Johnston also where they write *Any technique which successfully removes the contradiction ... we call a resolution, rather than a solution ..... a solution is a resolution together with a rationale*, I can concisely claim that in order to overcome logical paradoxes what we need is a solution, not a resolution.

18.3. The basic presupposition is that no contradiction can be derived from coherent assumptions through a coherent reasoning. Therefore a derived contradiction is unquestionable evidence that either the assumptions or the reasoning hide some incoherence (which an easy etymologic suggestion induces me to call "pseudologism"). I speak in singular because, once more, I join Goddard and Johnston also in emphasizing the similarity of structure characterizing all logical paradoxes: *this similarity of structure is fundamental. It justifies the intuitive view that piecemeal solutions are not solutions at all, and that there must be a single solution which applies to every paradox*. That is: the general solution of logical paradoxes is the clear explanation of the pseudologism they arise from. In this sense the solution must satisfy three prejudicial conditions, and precisely:

I^st^. To be powerful enough to enclose every known logical paradox (it must kill the central ganglion-cell of the octopus, not simply cut off a single tentacle).

II^nd^. Not to be so powerful to interdict argumentative techniques elsewhere useful and legitimate (it cannot kill the octopus by an explosion destroying the whole sea-fauna).

III^rd^. To be highly convincing on the ground of a general approach to logic (the denunciation of a solecism, once well understood, cannot be a baffling claim).

If I am not mistaking, the solution I propose satisfies these three prejudicial conditions. Here I expose it both through a formal and an informational approach. Yet there is a third and intuitively enlightening approach that, if Gods are benevolent, I will expose in a book specifically devoted to a representation of semantics.

18.4. First of all I recall
- that a dilemma is either closed (free-variable-free) or open (fee-variable-laden);
- that a closed dilemma is a question liable to two opposite answers;
- that an open dilemma is affected by an intrinsic lack of information forbidding any answer.

Until now I have spoken of (logical) paradoxes under the usual acceptation of the term, but henceforth I will speak of paradoxes to mean also what are usually called "antiparadoxes" (as for instance the Truthteller). My choice is justified by two pieces of evidence:

a) a dilemma whose two opposite solutions are self-confirming is not less puzzling than a dilemma whose two opposite solutions are self-contradictory

b) the same pseudologism leading to paradoxical dilemmas leads to anti-paradoxical dilemmas too, so that the same argument solves all of them.

18.4.1. Since my actual aim is to present the general solution in its most plain version, I do not pursue the widest approach, but the most accessible one. For instance only textual conversions are considered (that is: deictic conversions are neglected).

Obviously, no fixation (no effective conversion) of a free variable is possible if its antecedent too is affected by a free variable (in particularby the same free variable to fix). For instance, while the conversion of

Ava is glad but her mother is sad

is effective because the possibility to identify the glad person (Ava) entails the possibility to identify the sad person (Ava's mother),

she is glad but her mother is sad

is defective because the impossibility to identify the glad person (she who?) entails the impossibility to identify her mother.

Indexical dilemmas (more particularly: reflexive dilemmas) play a crucial role in the general solution of logical paradoxes. In fact such a solution can be summarized in the recognition that paradoxical dilemmas are (crypto) defective because, syntactically reasoning, the same free variable to fix occurs free also in the antecedent through which it ought to be fixed.

In other words paradoxical dilemmas are preposterous in the strictly etymologic acceptation of "preposterous" (*before after*) because their previous solution ought to represent the datum on whose grounds the same solution could be achieved. And the hypothetical procedure (if it were... then it ought to be....) by which the various paradoxes are characterized is exactly the consequence of their preposterousness, that is the vain attempt to overcome by a hypothetical postulation an intrinsic lack of information. In this sense Humphries's distinction between diagonal (constructive) and heterological (non-constructive) procedures concerns a secondary aspect of the matter.

18.5. The standard (extensional) procedure towards paradoxes can be schematized as follows
- to consider a set of individuals x
- to consider a set of sets Y
- to define a correspondence η between x and Y so that $Y = \eta x$
- to partition the pairs (x,Y) on the grounds of a relation ϕ between x and Y
- to consider the sets **A** and **Q** of those x such that $\phi\eta\, x$ and $\sim\phi\eta\, x$ respectively
- to select the individuals a and q such that $\eta a=\mathbf{A}$ and $\eta q=\mathbf{Q}$
- to realize that the dilemmas $|a\in\phi\eta a|$ and $|q\in\phi\eta q|$ (or indifferently $|a\notin\phi\eta a|$, and $|q\notin\phi\eta q|$) are defective.

The intensional procedure, once we adequate the notations above to an intensional context, is exactly the same; for instance the set **A** becomes the attribute (property) A and so on.

18.6. Let me apply this scheme to Richard's paradox (in one of its versions). Though for the sake of concision I define each set through the respective condition of membership, the discourse is even more easily applicable to sets defined through the list of their respective members.

Let ρ be a correspondence between natural numbers x and sets of natural numbers **Y**. Under ρ, say,

$\rho(4)$ is the set of even numbers, that is $\{y\colon \exists z(z+z=y)\}$

$\rho(7)$ is the set of quadratic numbers, that is $\{y\colon \exists z(z^2=y)\}$

$\rho(9)$ is the set of cubic numbers, that is $\{y\colon \exists z(z^3=y)\}$

$\rho(13)$ is the set of prime numbers, that is $\{y\colon \sim\exists wz(\sim(w=1)\&\sim(w=y)\ \&\ wz=y)\}$

and so on. Of course in order to ascertain whether a number is a member of a set we must ascertain whether such a number satisfies the condition of membership

(18.i) $x\in\{y\colon Py)\} \leftrightarrow Px$

then for instance, while 6 is a member of $\rho(4)$ because $\exists z(z+z=6)$, it is not a member of $\rho(7)$ because $\sim\exists z(z^2=6)$ and so on. In particular there are numbers (as 4 and 13) that are members of their set and numbers (as 7 and 9) that are not; it is then possible to define the (Richardian) set

(18.ii) $\{y\colon\ y\notin\rho(s)\}$

that is the set whose members are the numbers which do not belong to their respective set. In (ii) we could replace "s" with the usual "y", but such a replacement would be the result of a conversion, since the very condition of membership is not

(18.iii) $\notin\rho(y)$

(not belonging to the set assigned to a number to specify) but rather

(18.iv) $\notin\rho(s)$

(not belonging to the set assigned to **<u>itself</u>**, a condition obviously satisfied by the examples below). In other words, (18.iv) is the generic predicate and (18.iii) is simply its particularized conversion where y is the antecedent of the free reflexive variable. Once agreed that r is the number such that

$$\rho(r) = \{y\colon\ y\notin\rho(s)\}$$

the dilemma

$$|r\in\{y\colon\ y\notin\rho(s)\}|$$

is paradoxical since

$$r\in\rho(r) \leftrightarrow r\notin\rho(r)$$

follows from (18.i) and (18.iv).

The crucial achievement is realizing that the Richardian set is indexical (§16.8.1), so to call a set whose condition of membership is indexical. While the arithmetical property a number must (not) have in order (not) to be a member of an absolute set is always the same (for instance all the members of $\rho(4)$ are even, all the members of $\rho(7)$ are quadratic, all the members of $\rho(9)$ are cubic, all the members of $\rho(13)$ are prime) the arithmetical property a number must not have in order to be a member of $\rho(r)$ varies with the same number (for instance, while 7 is a member of $\rho(r)$ because it is **not quadratic**, *9* is a member of $\rho(r)$ because it is **not cubic** and so on). From a severely formal viewpoint I could also say: while

$$\exists z(z+z=\ldots)$$
$$\exists z(z^2=\ldots)$$
$$\exists z(z^3=\ldots)$$
$$\sim\exists yz(\sim(y=1)\&\sim(y=x)\&\ yz=\ldots)$$

are respectively the concatenations of primitive or previously defined symbols on whose grounds we form respectively $\rho(4)$, $\rho(7)$, $\rho(9)$ and $\rho(13)$, no analogous concatenation for $\rho(r)$ is possible.

18.6.1. Indeed if we should refer the Richardianity to ordered couples (§6.6), we could by-pass the indexicality; this heterodox approach will be analysed in §10 below, apropos of Grelling; here I insist on the orthodox approach. Under it, since the condition of membership depends on a relation (of non-membership) between any specific number and a set of numbers, such a set must be previously and perfectly known in order to answer any specific dilemma; otherwise our answer may be forbidden by a lack of information. Therefore where the condition of membership to $\rho(r)$ is sent back to the condition of (non)membership to the same set, we are facing a lack of information, a preposterous dilemma, a defective context, or better a cryptopreposterous dilemma, a cryptodefective dilemma, as this lack of information affects only two specific dilemmas: the paradoxical and the anti-paradoxical ones (in fact the argument holds identically for the anti-Richardian set, that is the set of numbers which are members of their respective set). From the formal viewpoint, the presence of the free reflexive variable, is not at all influenced by replacing "$\not\in$" with "$\in$" in (iv). Just because of this presence, while it is correct, for instance, assuming "**E**" to name $\rho(4)$ or "**C**" to name $\rho(9)$, assuming "**R**" to name $\rho(r)$ would be a misleading step because in a definition like

$$\in\mathbf{R} \leftrightarrow \not\in\rho(s)$$

the free variable occurring in the definiens does not occur in the definiendum . And as soon as we recognize that the correct formulation must be something like

(18.iv) $$\in\mathbf{R}_{\rho(s)} \leftrightarrow \not\in\rho(s)$$

the paradox vanishes. In fact, for instance, while $\mathbf{R}_{\rho(4)}$ is $\mathbf{R}_{\mathbf{E}}$, and $\mathbf{R}_{\rho(9)}$ is $\mathbf{R}_{\mathbf{C}}$, $\mathbf{R}_{\rho(r)}$ is $\mathbf{R}_{\mathbf{R}_{\rho(s)}}$ and in "$\mathbf{R}_{\mathbf{R}_{\rho(s)}}$" the free (reflexive) variable continues to occur. Let me insist on this central point. The dilemma

$$4\in\mathbf{R}_{\rho(s)}?$$

can be answered without any appeal to a hypothetical procedure (if 4 were …) because the conversion of the reflexive variable into

$$4\in\mathbf{R}_{\rho(4)}?$$

that is into

$$4\in\mathbf{R}_{\mathbf{E}}?$$

is effective (**E** is the previously and perfectly determined set of even numbers and the arithmetical structure of the number 4 (its evenness) is sufficient to entail that it is a member of **E**). On the other hand

$$r\in\mathbf{R}_{\rho(s)}?$$

is a (crypto)defective dilemma because its conversion leads to

$$r\in\mathbf{R}_{\mathbf{R}_{\rho(s)}}?$$

where the same reflexive variable continues to occur free, and the appeal to a hypothetical procedure (if *r* should belong …) is only the already mentioned vain attempt to escape an intrinsic lack of information; *r* can neither belong to the Richardian set nor to its complementary because these sets are defined in such a way that their condition of membership is defective in exactly two particularizations (such sets are locally fuzzy, to follow a terminology Fine ought to appreciate).

Therefore I partially agree with Goedel's intuition where he writes (1944, p.150): *it might even turn out that it is possible to assume every concept to be significant everywhere except for certain "singular points" or "limiting points", so that the paradoxes would appear as something analogous to dividing by zero*. What I cannot accept is his excess of generalization (*every concept*) because only where a pretence of auto-conversion is involved we meet such singular points. Trivially: if the barber comes from another village, no paradox arises.

18.6.2. Russell's paradox, whose classical extensional version can be sketched as follows

$$x\in y \leftrightarrow x\not\in x$$
$$y\in y \leftrightarrow y\not\in y$$

is nothing but a simplified version of the above scheme and enters into the same solution (*y* is an indexical set whose intension is $\not\in s$). In other words. The formally correct definition

$$x\in c_x \leftrightarrow x\not\in x$$

(that is: $x \in c_s \leftrightarrow x \notin s$)
overcomes any impasse, since

$$c_x \in c_{cx} \leftrightarrow c_x \notin c_x$$

(that is: $c_s \in c_{cs} \leftrightarrow c_s \notin c_s$)
far from being contradictory, expresses an unquestionable truth ("*c*" is a sort of negation).

18.7. I dwell on a collateral but perhaps intriguing consideration. Once some desperate *ad hoc* proposals (such as banning *tout court* reflexive predicates from the language) are rejected, and once some marginal differences between Poincaré's and Russell's positions are neglected, impredicativity is the only general solution of logical paradoxes till today suggested. According to it, *any entity defined by an expression which contains a bound variable must be excluded from the values of this variable*. Unfortunately such a solution is too expensive. *This vicious circle principle ... is in danger of mutilating rather than purifying mathematics* (Beth 1959, p.499). And precisely with the aim of overcoming this heavy difficulty a distinction has been introduced (Hintikka 1956, p. 244) between an *innocent* and an *insidious impredicativity*. According to Beth's same examples,

$$\forall y(xy=x \ \& \ yx=x \ \& \ \exists z(yz=x \ \& \ zy=x))$$

(which defines the unitary element of a group) and

$$\forall y(\exists z(z \in m \ \& \ z>y) \leftrightarrow (y<x))$$

(which defines the least superior border of a set *m* of natural numbers) and

$$\forall x(x \in n) \leftrightarrow \forall m((1 \in m \ \& \ (y \in m \rightarrow (y+1) \in m)) \rightarrow x \in m)$$

(which defines the set *n* of natural numbers) are all innocent (and theoretically precious) impredicative definitions.

On the other hand

(18.v) $\forall x(x \in y \leftrightarrow \sim(x \in x))$

(which defines the set of sets which are not members of themselves), is an insidious impredicative definition.

But from our viewpoint it is easy to remark that only (18.v), once reformulated through the help of a reflexive variable in order to underline the crucial passage, would involve an auto-conversion. This is the deep reason why (Hintikka, ibidem) *paradoxes cannot be formulated in the terms of the 'innocent' impredicativities.*

Other proposals (as Hintikka's strongly exclusive interpretation of quantifiers) succeed in forbidding such auto-conversion but (at least in my opinion) fail in explaining why and where such a strongly exclusive interpretation is necessary.

18.7.1. If we intend to preserve the term "impredicativism" to mean the pseudologism the logical paradoxes are born by,

No indexical free variable can be converted by a free-variable-laden antecedent

is the very rule that bans any impredicativism. But indeed, more than a specific addictive rule, it is simply an obvious requirement telling us that if we violate it, we get an expression where a free variable continues to occur, and treating a free-varable-laden expression as a free-variable-free one is unquestionably incoherent.

In this sense the informational approach does not require any addictive prescription besides the respect of coherence, that is, in our case, not defining an indexical quantity and treating it, although implicitly, as if it were absolute. Neither a paradoxical dilemma is self-contradictory nor an anti-paradoxical dilemma is self-legitimating. Both of them are (crypto)defective because both of them hide a lack of information.

18.8. Since I do not know Grelling's paradox in his original version, I quote the Britannica (whose terminological choices will be adopted in the following discussion too).

> Let us classify the adjectives of the English language as to whether they are self-applicable or non-self-applicable. An adjective is self applicable if it has the property it expresses; *e.g.* the adjective "short" is self-applicable since it is a short word, but "long" is non-self-applicable since it is not a long word. Every adjective is either self-applicable or non-self-applicable, and cannot of course be both. Which is the case for the adjective "non-self-applicable"? Suppose that it is self-applicable. Then it has the property it expresses, i.e. it is non-self-applicable, contrary to our supposition. Thence *it is non-self-applicable*. This means that it does not have the property it expresses, the property on non-self-applicability. But this is just another way of saying that *it is not non-self-applicable*. We have again arrived at two contradictory results.

In order to formalize the solution in the simplest way I agree upon the following symbology:

- *"x", "y"* ...variables ranging over adjectives

and in particular

- *"b", "m", "f"* and *"u"* "blepharospastic", "monosyllabic", "fresh" and "useless" respectively
- *"a"* and *"q"* "self-applicable" and "non-self-applicable" respectively
- *X, Y, B, M* et cetera properties respectively adduced by *"x", "y", "b". "m"* et cetera.

I underline that the use of individual variables and individual constants as *"x", "b"* et cetera for adjectives is not at all an abusive license, owing to the metalinguistic viewpoint from which we observe them. That is: *L*-adjectives are named by *ML*–substantives (§2.5: substantivizing effect of quotation marks).

As soon as we (provisionally) agree to define the self-applicability and the non-self applicability by

(18.vi) $A(x) \leftrightarrow X(x)$
(18.vii) $Q(x) \leftrightarrow \sim X(x)$
we derive the contradictory
(18.viii) $Q(q) \leftrightarrow \sim Q(q)$
by particularization of (18.vii) on *q* (for the sake of concision henceforth I set aside (18.vi) and reason only on (18.vii)).

Though here too the solution depends on the logical impossibility of promoting an indexicality by an auto-conversion, I enter into details, reminding the reader of the detailed example proposed in §16.6.

18.9. Let *(x,Y)* be an ordered couple where the adjective *x* and the property *Y* are casually joined. By definition a couple is concordant *(C)* iff *Y(x)* and discordant *(D)* iff ~(*Y(x))*.

Indeed a more articulate analysis (§9.5.1) ought to distinguish between properly discordant couples as for instance *(b,M)* where the "~" occurring in "*~M(b)*" means an oppositive negation, and improper (or improperly discordant) couples as for instance *(m,B)* where the "~"occurring in "*~B(m)*" means an exclusive negation. Yet I am pleased with the simple opposition between concordant and discordant couples because
- the quoted version of the paradox complies with such a simplified assumption (*every adjective is either self-applicable or non-self-applicable*)
- the same assumption is the usual one (cf. for instance Church in D.D.Runes *The Dictionary of Philosophy* under "(Logical) Paradoxes")
- the argument can be easily extrapolated to a further distinction between proper and improper discordance.

Then
(18.ix) $C(x,Y) \leftrightarrow Y(x)$
and
(18.x) $D(x,Y) \leftrightarrow \sim(Y(x))$
symbolize the above definitions (here too for the sake of concision I set aside (18.ix)). Just as (18.x) defines a property pertaining to ordered couples
(18.xi) $D_Y(x) \leftrightarrow \sim(Y(x))$
defines the *Y*-discordance, a (relational) property that, obviously, pertains to adjectives.

In order to answer any specific dilemma concerning the *Y*-discordance of a given adjective it is sufficient to particularize (18.xi). For instance, to ascertain whether *b* is *M*-discordant we proceed as follows:

$D_M(b)?$ $\sim(M(b))?$

and since we know what "monosyllabic" means and how many syllables "blepharospastic" is formed by, we can conclude

$\sim(M(b))$ $D_M(b)$

that is the *M*-discordance of the adjective under scrutiny. Of course should we start from

$C_M(b)?$

the conclusion would be the same (that is $\sim C_M(b)$), as the dilemma is the same.

18.9.1. Since a correspondence η exists between an adjective and the property itmeans ($\eta=\sigma$)

$X=\eta x$

therefore the assumption
(18.xii) $Y=X$
transforms the discordance into the non-self-applicability; in fact, under (18.xii) the second member of any ordered couple is just the property expressed by the respective first members. Nevertheless **the 'binary nature' of every couple is not compromised by the possibility of 'computing' its second member in function of its first**. Though it pertains to adjectives, the (non)self-applicability continues depending on the expressed property, too. For instance if we interpret "fresh" as a synonymous of "new", "fresh" is non-self-applicable for it is an old adjective, but if we interpret "fresh" as a synonym of "active", it is self-applicable.

In other words. Exactly as "he" is a precise and constant indexical substantive which, in different contexts, is converted into different absolute substantives (as "Tom", "Bob" and so on), "non-self-applicable" is a precise and constant indexical adjective which, in different contexts, is converted into different absolute adjectives (as "polysyllabic", "useful" and so on). In this sense, just as we say that "he" is a pronoun (a pro-noun), we can say that "non-self-applicable" is a proadjective (a pro-adjective). And in fact where ascribed to "monosyllabic" the non-self-applicability is the polysyllabicity ("non-self-applicable" is converted into "polysyllabic"), where ascribed to "useless" the non-self-applicability is the usefulness ("non-self-applicable" is converted into "useful") and so on. Therefore **the presence of a free variable in its symbolization is a formal must**, exactly in order not to mistake the typographical invariance of the adjective for the invariance of the adduced property. Informally, such a tramp is evident in the version replacing “non-self-applicable” and “self-applicable” with “heterologic” and “homologic”, because, contrary to “self”, “hetero” and “homo” tend to conceal their role of variables.

This conclusion is formally confirmed by a comparison between (18.vii) and

$D_X(x) \leftrightarrow \sim(X(x))$

(drawn from (18.x) through (18.xii)) Realizing that *“Q”* is an elliptically incorrect notation for *“Q_X”* is realizing the impossibility to obtain an effective auto-conversion, since manifestly in *“Q_{QX}”* the free variable continues occurring free. Furthermore it is also realizing why, on the contrary,

(18.xiii) is “blepharospastic” (non)self-applicable?

or

(18.xiv) is “monosyllabic” (non)self-applicable?

And so on are effective dilemmas (no free variable survives to the conversion of *“Q_X”* into *“Q_B”* or into *“Q_M”* and so on). Finally it is also realizing why Grelling’s dilemma is crypto-defective; because all the other analogous dilemmas, as (18.xiii), (18.xiv) and so on are effective (non-preposterous), that is because, save two exceptions (two *singular points*), in a scheme like

is …... (non)self applicable?

replacing the dots with the name of an adjective bears a non-defective dilemma.

18.9.2. Another strong argument supporting the indexicality of (non-)self-applicability runs as follows. The (im)properness of any proposition obtained by the attribution of an absolute property to an adjective does not depend on the adjective. For instance, since

“useless” is monosyllabic

is (false but) proper, we can substitute “useless” with whatever other adjective without losing the properness of the resulting proposition (which for instance is true (then proper) if the substitutor is “fresh”, false yet anyhow proper if the substitutor is “blepharospastic” et cetera). On the other hand, though

“useless” is non-self-applicable

is another (false but) proper proposition, if the substitutor is “blepharospastic” we get an improper proposition and if the substitutor is “fresh” we get a proper proposition which will be true or false in compliance with the acceptation of “fresh”. This deep discrepancy depends just on the fact that the meaning of “non-self-applicable” (the property it expresses) varies with the adjective it is attributed to; such a deep discrepancy, then, depends on the deep discrepancy between absolute (invariant) and indexical (variant) properties.

18.10. If we modify the original version of Grelling’s paradox by making *Q* a property pertaining to ordered couples as *D*, that is if we assume by definition

(18.xv) $Q(x,X) \leftrightarrow \sim(x(x))$

the same *Q* is no longer an indexical property ((18.xv) is a formally correct formula as (18.x)). Yet no contradiction can be derived from (18.xv); in fact the particularization of *x* on *q* (therefore the particularization of *X* on *Q*) leads to

(18.xvi) $Q(q,Q) \leftrightarrow \sim(Q(q))$

and (18.xvi), far from being contradictory, allows us to deepen the analysis by punctuating the two possible readings of “$\sim$”. In the oppositive reading, since *Q* pertains to couples, *Q(q)* is improper, and the particularization is then illegitimate. In the exclusive reading, precisely because *Q(q)* is improper, the second member of (18.xvi) is true. This means that, under this reading,

$Q(q,Q)$

is true (the couple is non-self-applicable); a conclusion which agrees perfectly with our intuition, for actually *Q* (by definition a property pertaining to couples) cannot be properly ascribed to an adjective, therefore *(q,Q)* is a non-self-applicable couple.

18.10.1. A willing reader might be tempted to revive the paradox by a definition like

(18.xvii) $Q(x,X) \leftrightarrow \sim(x(x,X))$

where actually both variables of the definiens occur also in the definiendum. A misleading temptation, because, contrary to (18.xvi), in (18.xvii) both the definiens and the definiendum have the same argument (that is *(x,X)*); then, put shortly, (18.xvii) states an equivalence between *Q* and $\sim X$; but of course, as *X* is an indexical property which varies with *x*, its indexicality entails that *“Q”* is an incorrect (and misleading) notation. In other words, (18.xvii) falls into the same formal omission affecting (18.vii).

18.11. The Liar (the Truthteller) too is born by the preposterousness of the paradoxical dilemma. Its peculiarity depends only on the different status of the objects the reflexive variable ranges over; while Richard refers to numbers, Russell to sets and Grelling to adjectives, Liar and Truthteller refer to sentences, that is to objects whose syntactical status is the same as that of the (paradoxical) dilemma under scrutiny. Indeed it would be better to reason about propositions, yet in order to adequate the following discourse to the current terminology I accept not only to reason about sentences but also to speak of false and true sentences though I ought to speak of fallacious and veracious sentences (the only caution is the use of "*FL*" and "*VR*" instead of "*F*" and "*T*" respectively). Anyhow a stricter approach is sketched in §18.11.7.

The canonical formulation of the Liar, concisely, runs as follows:

(18.xviii) this sentence is false

is self-contradictory because if it were true, it would state its falsity, so it would be false, and vice versa.

Reciprocally (Truthteller)

this sentence is true

is anyhow self confirming because et cetera.

Once recalled that, according to Tarski's approach,

(18.xix) $VR(\text{“}Y(x)\text{”}) \leftrightarrow Y(x)$

and

(18.xx) $FL(\text{“}Y(x)\text{”}) \leftrightarrow \sim Y(x)$

are the conditions for truth and falsity, I organize the solution through five progressive examples.

18.11.1. First example (standard context). The assignation of an alethic value to

(18.xxi) Ava is blonde ($B(a)$)

that is the solution of the respective dilemma

($B(a)$)?

runs as follows:

$VR(\text{“}B(a)\text{”})?$

$VR(\text{“}B(a)\text{”}) \leftrightarrow B(a)$

$B(a)?$

or equivalently

$FL(\text{“}B(a)\text{”})?$

$FL(\text{“}B(a)\text{”}) \leftrightarrow \sim B(a)$

$\sim B(a)?$

(the two opposite questions are equivalent because, obviously, answering one of them is answering the other, that is because their common dilemma depends on the same collative datum, just represented by the piece of information concerning Ava's hair). And since Ava is raven haired,

(18.xxii) $\sim B(a)!$

is the acquirement we draw from the sight of her hair, so the same (18.xxii) allows us to conclude that $\sim VR(\text{“}B(a)\text{”})$ or equivalently that $FL(\text{“}B(a)\text{”})$.

18.11.2. Second example (metalinguistic context). The assignation of an alethic value to

(18.xxiii) "Ava is blonde" is (printed with) green (ink) $G(\text{“}B(a)\text{”})$

runs analogously as follows:

$VR(\text{“}G(\text{“}B(a)\text{”})\text{”})?$

$VR(\text{“}G(\text{“}B(a)\text{”})\text{”}) \leftrightarrow G(\text{“}B(a)\text{”})$

$G(\text{“}B(a)\text{”})?$

and the collative datum (drawn from the sight of the object sentence (18.xxiii) speaks of) is

(18.xxiv) $\sim G(\text{“}B(a)\text{”})$

(the object sentence (18.xxiii) speaks of is not printed with green ink) therefore (18.xxiv) allows us to conclude that $\sim VR(\text{“}G(\text{“}B(a)\text{”})\text{”})$ or equivalently that $FL(\text{“}G(\text{“}B(a)\text{”})\text{”})$.

18.11.3. Third example (auto-referential context). The assignation of an alethic value to

(18.xxv) this sentence is green ($G(s)$)

runs analogously as follows:

$VR(\text{“}G(s)\text{”})?$

$VR(\text{“}G(s)\text{”}) \leftrightarrow G(s)$

$G(s)?$

that is, by conversion of the reflexive variable (whose antecedent is the whole (18.xxv))

(18.xxvi) $G(\text{“}G(s)\text{”})?$

and the collative datum (drawn from the sight of the sentence (18.xxv) speaks of, that is the same (18.xxv)), is

(18.xxvii) $\sim G(\text{“}G(s)\text{”})$

(the sentence (18.xxv) is not printed with green ink) therefore (18.xxvii) allows us to conclude that $\sim VR(\text{“}G(s)\text{”})$. Of course the mentioned conversion is effective because what occurs in (18.xxvi) is not a free (reflexive) variable, but **the name of a sentence where the same variable occurs**, that is because the same variable is mentioned, not used, and obviously the name of a variable is a constant exactly as the name of a constant); furthermore the just ascertained possibility of solving (18.xxvi) is the best evidence that it is a free-variable-free dilemma.

Incidentally. Though (18.xxv) is unquestionably auto-referential, the alethic procedure did not present any difficulty. The proposal of overcoming the Liar by forbidding auto-referential sentences is then, at the very least, too severe an intervention (I recall the condition IInd of §18.3).

18.11.4. Fourth example (alethic context). The assignation of an alethic value to
(18.xxviii) "this sentence is green" is true (*VR("G(s)")*)
runs analogously:

VR("VR("G(s)")")?
VR("VR("G(s)")")↔VR("G(s)")
VR("G(s)")?

the collative datum is then represented by the result of a collation. In order to achieve this datum the context must allow us to perform such a collation, and in order to perform such a collation we must previously know both *correlata*, that is the proposition adduced by the object sentence within quotation marks in (18.xxviii) and the cognition drawn from the sight of the ink used to print the same object sentence; since we know both of them, we can actually achieve the result, and since it is anti-collative (since our statute acquires the (meta)cognition that the object proposition is rejected by the object cognition), the collative (meta)datum

VR("G(s)")

allows us to conclude that

FL("VR("G(s)")"

(actually it is false that the object sentence (18.xxviii) speaks of is true).

18.11.5. The fifth example (auto-referential alethic context) is the Truthteller. The assignation of an alethic value to
(18.xxix) this sentence is true (*VR(s)*)
runs analogously:

VR("VR(s)")? *VR("VR(s)") ↔ VR(s)* *VR(s)?*

and here too, since (18.xxix) concerns an alethic predicate just as (18.xxviii), the collative (meta)datum ought to be represented by the result of a collation. But here the context does not allow us to perform such a collation because we cannot know the second *correlatum*; in fact it ought to be represented by the result of the same collation under scrutiny (something like Baron Münchhausen pulling himself out of a swamp by his own hair).

In other words. While
(18.xxx) *VR("G(s)")?*
is an effective dilemma because the typographical aspect of the object sentence allows us to ascertain its (non)greenness (allows us to know the homologous cognition), contrary to (18.xxx)

VR(s)?

that is, by conversion,

VR("VR(s)")?

is a defective (preposterous) dilemma because the intrinsically relational nature of alethic predicates forbids us to ascertain them on the only ground of the sentence (of the proposition) under scrutiny.

Reciprocally (yet identically), if we start from (18.xviii), we get

FL("FL(s)")?
FL("FL(s)") ↔ ~FL(s)
~FL(s)?

therefore here too the collative datum ought to be represented by the solution of the dilemma under scrutiny.

Both Liar and Truthteller are affected by the same and congenital lack of information.

18.11.6. The 'double face' version of the Liar ("the next sentence is true" where the next sentence is "the preceding sentence is false") is immediately reducible to the classical version. In fact the sentence stating that its next sentence is true, states that the sentence preceding its next sentence is false, then it states its own falsity. Thus the preposterousness is fully maintained.

18.11.7. In order to show how preposterousness can be reduced to an attempt of auto-conversion, let me submit Tarski's condition to a deeper analysis.

Alethics is an intrinsically relational doctrine because, roughly speaking, it concerns some relations between sentences and facts (better: between propositions and cognitions). A (declarative) sentence x singles out a proposition σx, and a proposition singles out a homologous cognition $\psi\sigma x$ verifying or falsifying it (strictly: a proposition singles out a k-cognition k-verifying or k-falsifying it); therefore two opposite propositions have the same homologous cognition. Just as the above approach (which adopts the canonical viewpoint according to which alethic predicates pertain to sentences) leads to homologous couples $(\sigma x, \psi\sigma x)$, the (correct) viewpoint (according to which alethic predicates pertain to propositions) would lead directly to couples $(y, \psi y)$. Anyhow a basic distinction opposes the concordant couples C defined through

(18.xxxi) $C(\sigma x,\ \psi\sigma x) \leftrightarrow (\sigma x = \psi\sigma x)$
(that is: $C(\sigma x,\ \psi\sigma x) \leftrightarrow (\sigma x = \psi s)$)
and discordant couples *D* defined through
(18.xxxii) $D(\sigma x,\ \psi\sigma x) \leftrightarrow (\sim\sigma x = \psi\sigma x)$
(that is: $D(\sigma x,\ \psi\sigma x) \leftrightarrow (\sim\sigma x = \psi s)$)
(henceforth, for the sake of concision, I neglect (18.xxxii).

To replace (18.xxxi) with
(18.xxxiii) $C(\sigma x) \leftrightarrow (\sigma x = \psi s)$
would be a formal and substantial abuse. In fact
- formally, in (18.xxxiii) "*C*" becomes an abbreviation of "$=\psi s$", then a free reflexive variable disappears;
- substantially, (18.xxxiii) would make the concordance between a proposition and its homologous cognition a property of the same proposition (which obviously is not, since the homologous cognition is an independent and determinative datum).

Indeed, once more, what we can correctly derive from (18.xxxi) is
(18.xxxiv) $C_{\psi s}(\sigma x) \leftrightarrow (\sigma x = \psi s)$
where the indexicality of the predicate "concordant with **its** homologous" is correctly witnessed by "$C_{\psi s}$" (unquestionably if a homologous couple is concordant, the proposition is concordant with its homologous cognition). But (18.xxxi) and (18.xxxiv) show that the context is strictly analogous to Richard's and Grelling's ones: as $\sigma x = \psi s$ is the condition of truth, the indexical predicate $C_{\psi s}$ is nothing but the predicate of truth, and any attempt of auto-conversion for the free reflexive variable is devoted to an even formal failure.

18.11.8. Let me resume by particularizing the *"Y(x)"* of (18.xix) on some already proposed sentences:

"B(a)"	*"G(s)"*	*"VR("G(s)")"*	*"VR(s)"*
VR("B(a)")?	*VR("G(s)")?*	*VR("VR("G(s)")")?*	*VR("VR(s)")?*
(Tarski's condition)			
VR("B(a)")↔B(a)	*VR("G(s)")↔G(s)*	*VR("VR("G(s)")")↔ VR("G(s)")*	*VR("VR(s)")↔ VR(s)*
B(a)?	*G(s)?*	*VR("G(s)")?*	*VR(s)?*
	by conversion		
	G("G(s)")?		*VR("VR(s)")?* Circularity:
by acquirement of the respective cognition			the grounding cognition ought to be
		~G(s)	preposterously represented by its
~B(a)	*~G("G(s)")*	*~VR("G(s)")*	same solution.
	~G(s)		
~VR("B(a)")	*~VR("G(s)")*	*~VR("VR("G(s)")")*	*?????*

The correct conclusion is once more the same: neither the Liar dilemma is self-contradictory nor is the Truthteller anyhow self-confirming; their congenital lack of information forbids them to be true or false.

In two words. The alethic value of a proposition does not depend only on the same proposition, since it concerns a link between such a proposition and its homologous cognition. Therefore an alethic predicate can be properly ascribed to a proposition iff the homologous cognition is non-preposterously attainable.

18.12. Such a conclusion, generally speaking, shows that to ground an apagoge (a *reductio ad absurdum*) on a defective dilemma is a totally spurious argument. In fact if
$|h|_k?$
is a defective dilemma, deriving
$T_k(\sim h)$
from
$\sim T_k(h)$
is a quite abusive inference because, just owing to the defectivity of the dilemma, neither *h* nor $\sim h$ can be *k*-true.

18.12.1. The clear and sound assimilation of the above evidence entails that Cantor's celebrated proof is not at all a proof. His argument can be sketched as follows. Let *m* be a numerable set and $\mathcal{P}m$ its power set, that is the set of its subsets. A one to one correspondence κ between the members *x* of *m* and the members *y* of $\mathcal{P}m$ cannot exist. In fact should it exist, then a *b* and a *c* would exist such that
$\kappa b = c = \{x:\ x \notin \kappa(x)\}$
and the dilemma
$b \in c?$
would lead to a contradiction.

The analogies with Richard's paradox are more than evident (*c* is an indexical set and an indexical dilemma becomes decidable only after its conversion into an absolute one et cetera). The preposterousness too is manifest: in order to decide whether *b* does belong to *c* (so getting an effective conversion of the free reflexive variable) we should previously know whether *b* does or does not belong to *c*. So, once more, we have to remark that the worst logical puzzles ensue from the misrecognized presence of a free reflexive variable.

18.12.1.1. This confutation, of course, does not show that Cantor's theorem is misleading; it only shows that his claim is a simple proposal, thus offering a theoretical support to the opinion of some scholars who reject Cantor's whole theory of transfinites on mere humoral bases.

18.12.2. A similar situation concerns Cantor's lemma of diagonalization (Shoenfield 1967, §6.8). It is nothing but the conclusion that no indexical predicate is absolute. What is wrong in a definition like

$$Q(x) \leftrightarrow \sim P_x(x)$$

that is like

(18.xxxv) $Q(x) \leftrightarrow \sim P_s(x)$

is the abusive elimination in "*Q*" of the free variable occurring in the predicate of the definiens. And as soon as we realize that the formally correct definition ought to be something like

(18.xxxvi) $Q_s(x) \leftrightarrow \sim P_s(x)$

we realize that the only (and highly questionable) way to defend (18.xxxv) is to agree that an *"s"* is elliptically included in *"Q"* (as for instance a "self" is elliptically included in "heterologic"); but under this agreement *"Q"* becomes a predicative variable which, obviously, cannot be identified with any absolute predicate *"P_b"* (with reference to the example of §16.6, *b* would be the exact homologous of Utopolis). In other words, if we start from the correct (18.xxxvi), no conversion can lead to

$$Q_s(x) \leftrightarrow \sim P_b(x)$$

because the substitution of "*b*" to "*s*" must involve the reflexive variable at the first member, too; the right substitution leads to

$$Q_b(x) \leftrightarrow \sim P_b(x)$$

where Q_b is precisely the absolute predicate corresponding to that conversion of the indexical one. And as far as I know, no contradiction is derivable from (18.xxxvi).

18.13. Another simple application of the distinction between indexical and absolute predicates concerns Thomson's Little Theorem (Butler 1962, p.94). It claims that, if ρ is a relation defined on a set *m*, no member *x* of *m* can be in ρ–relation with all the *m*-members *y* which are not in ρ–relation with themselves. But once we symbolize the theorem in

$$\sim\exists x(x \in m \& \forall y \in m(\rho(x,y) \leftrightarrow \sim\rho(y,y)))$$

it is immediately evident that supposing the existence of a *q* such that

$$\rho(y,q) \leftrightarrow \sim\rho(y,s)$$

is identifying the indexical predicate

$$\sim\rho(s)$$

with an absolute predicate

$$\rho(q)$$

(here too I could evoke Utopolis). And here too I could underline that the occurrence of "~" is of no theoretical moment, because also

$$\rho(s) \leftrightarrow \rho(q)$$

is an illegitimate assumption (a Thomson's Little Counter-theorem is not less valid, although its violation does not lead to any direct formal contradiction).

18.14. A further step. Generally speaking, an assumption like

(18.xxxvii) $\exists x \forall y(x \varepsilon y \leftrightarrow \Phi x)$

is misleading if the only condition for the particularizations of

$$\Phi x$$

(just in order to avoid that *"y"* be abusively bound) is that the same *"y"* cannot occur free in Φ (without inverted commas, I am speaking of an object predicative expression, that is of the concatenation of object symbols named by "Φ"). In fact under that only condition no distinction is possible between absolute predicates, where no free variable occurs, and indexical (reflexive) predicates where a "*x*", (that is a "*s*") occurs. But (18.xxxvii) is exactly the axiomatic scheme of abstraction the current set theories are grounded on. May it be a mere casualty that all the formal contortions the set theorists are constrained to adopt with a view to avoiding contradictions are born by indexical predicates, or better by the mentioned lacking distinction?

18.15. Hoping that a surrealistic joke is welcome (or at least tolerated) I take advantage of a defective apagoge to prove what I modestly call “Gandolfi’s Great Theorem”: it simply states that our universe is incoherent. Here is the proof.

Let me consider the ordered couples of natural numbers. On the one hand, as the simple tabulation

⟨0,0⟩	⟨0,1⟩	⟨0,2⟩	⟨0,3⟩	⟨0,4⟩
⟨1,0⟩	⟨1,1⟩	⟨1,2⟩	⟨1,3⟩	⟨....⟩
⟨2,0⟩	⟨2,1⟩	⟨2,2⟩	⟨....⟩	
⟨3,0⟩	⟨3,1⟩	⟨....⟩		

is diagonally exhaustible (firstly the only couple whose sum is zero, secondly the two couples whose sum is 1, thirdly the three couples whose sum is 2 and so on), such couples must be numerable.

On the other hand, since they are numerable, we can establish a one to one correspondence θ between them and the binary numerical relations definable in English, which surely are numerable. Thus, as

> the members of an ordered couple are linked by the Gandolfian relation *GN* iff they are not linked by the relation θ-corresponding to the same couple

is one of such definitions, there is a couple $\langle x^\circ,y^\circ\rangle$ such that $GN=\theta\langle x^\circ,y^\circ\rangle$. And as soon as we consider

(18.xxxviii) $GN\langle x^\circ,y^\circ\rangle$

we fall into a contradiction.

Here too the impasse follows from a *reductio ad absurdum* focused on an inconvertible indexical dilemma (“the relation θ-corresponding **to the same couple**”). And the real target of so irreverent a Great Theorem is just to emphasize the essential presupposition which every sound *reductio* is grounded on, that is the non-defectivity of the respective dilemma. In other words, (18.xxxviii) is a misleading notation since *GN* is an indexical relation, and as soon as we indicate it by “GN_s” we realize that an auto-conversion is a pseudo-promotion to absoluteness, because it leads to an expression where the reflexive variable continues occurring.